Metallic and insulating behaviour in an exactly solvable random matrix model

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Abstract. We propose a novel random transfer matrix model for quantum transport in disordered systems. The model is exactly solvable in the sense that arbitrary n-point correlation functions of the eigenvalues can be obtained in terms of known orthogonal polynomials, and the conductance is a simple linear statistic of these eigenvalues. The model exhibits qualitative deviations from the universal properties normally associated with random matrices, and we observe that such deviations may naturally describe the differences in the distribution of conductance in the metallic versus insulating regimes. In particular, by varying a single parameter q, we recover the metallic regime with Ohm's law and universal conductance fluctuation in the limit $q = 1$, and the well known log–normal distribution of conductance for one-dimensional insulators in the opposite limit $q \ll 1$. We argue that in this model the metal–insulator transition is related to the qualitative change in the eigenvalue density and the associated opening of a gap in the density at the origin.

1. Introduction

In a recent letter [1], we proposed an exactly solvable random matrix model for electronic transport in disordered quantum conductors. The model is based on the maximum-entropy *ansatz* for random transfer matrices that describe quantum transport in disordered mesoscopic systems [2]. This model is motivated by known exact results in one dimension as well as recent numerical studies [3] that confirm the general validity of such an *ansatz*, where the eigenvalues of a certain combination of the transfer matrices repel each other logarithmically, but remain bounded by a single-particle confining potential whose functional form is quite insensitive to disorder, size or dimension. Distribution of physically observable conductance is then related to the distribution of eigenvalues of the above-mentioned random matrices. Such a mapping, if it exists, will allow us to address the issue of how the distribution of conductance changes as one goes from the metallic to the insulating side of a system (e.g. by increasing the disorder), as opposed to considering the average of conductance only, which is known to be insufficient for mesoscopic systems [2]. We showed that for a qualitatively realistic model, n-point correlation functions of the eigenvalues can be written down exactly (in terms of known orthogonal polynomials), which would not be possible in a conventional approach to quantum transport [4].

While random matrix models have been extensively studied in nuclear physics [5] and string theories [6], they have the common feature that the eigenvalues (real) are extended from $-\infty$ to $+\infty$ and the confining potential is a polynomial (related to Gaussian distribution of random matrix elements of the Hamiltonian of a complex nucleus, or discretization of random surfaces in a string theory). Similar models have been suggested for the Hamiltonian of disordered systems as well [7]. This is essentially on the basis of the universality of the conductance fluctuations in the metallic regime, which is very similar to the fluctuations in the energy levels of complex nuclei and these are well described within the standard random matrix models. All such models are known to have some universal properties, namely that the density of eigenvalues at the origin (small eigenvalues) scales with the number of eigenvalues, and the distribution of the spacing between nearest-neighbour eigenvalues follow the 'Wigner surmise' [5], a power-law behaviour for small spacings.

The problem of quantum transport in disordered systems, described in terms of transfer matrices, has led us on the other hand to consider a random matrix model where the eigenvalues are positive semi-definite and the confining potential behaves as a square of logarithm for large eigenvalues. Such a random matrix model has not been considered before. In our short note we reported considering an exactly solvable model of this type which seems to have interesting deviations from the universal properties normally associated with random-matrices, and observed that such deviations may naturally describe the differences in the distribution of conductance in the metallic versus insulating regimes. Although the model we chose may be an oversimplification of the actual model that might describe the problem of metal–insulator transition in a real physical system, it is nevertheless interesting to investigate such an exactly solvable model in some detail in order to understand certain qualitative features of how the conductance distribution evolves from metallic to insulating behaviour.

However, although the *n*-point correlation functions for the eigenvalues for our model can be written down explicitly in terms of a family of orthogonal polynomials as in the standard random matrix approach, the polynomials turn out to be the *basic* or so-called *q*-Laguerre polynomials of Hahn [8], and are not familiar in the physics community. In fact these have been investigated in some detail only recently [9], and many properties, including the asymptotic behaviours are either not known yet, or have only recently been obtained in connection with our present work [10]. We therefore present in this paper some details of the derivation of our results reported earlier [1]. We include in appendix A a complete asymptotic series for the generalized *q*-Laguerre polynomials of large degree that was not available before.

2. Description of the framework

An N-channel conductor can be described by a $2N \times 2N$ transfer matrix T which relates the left-hand flux to the right-hand flux. The doubly degenerate eigenvalues x_i $(0 \leqslant x_i \leqslant \infty, \ i = 0, \ldots N - 1,)$ of the matrix $\mathsf{X} = \frac{1}{4}[\mathsf{T}^\dagger \mathsf{T} + (\mathsf{T}^\dagger \mathsf{T})^{-1} - 2\mathsf{I}]$ are then related to the two-probe conductance g as [11]

$$g = \sum_{n=0}^{N-1} \frac{1}{1 + x_n}. \tag{1}$$

Note that a large density of small eigenvalues will correspond to a metal, while exponentially large eigenvalues will characterize an insulator. The number of eigenvalues N depends on the cross-sectional area of the conductor, so that $N \propto L^{d-1}$ for a conductor of size L^d in d dimensions.

The fundamental notion of the random matrix approach is that the ensemble of all random $\mathbf{X}$ matrices consistent with given symmetries (current conservation, time reversal etc) subect to some physical constraint (e.g. given average conductance) has a distribution of eigenvalues that can be written quite generally in the form [2]

$$P\{x_0,\ldots,x_{N-1}\} = \prod_{m<n,0}^{N-1} |x_m - x_n|^\beta \prod_{k=0}^{N-1} \exp[-V(x_k)]. \tag{2}$$

The distribution of conductance g can be obtained from $P\{x_i\}$ as follows:

$$P(g) = \left\langle \delta\left(g - \sum_{i=0}^{N-1} \frac{1}{1+x_i}\right)\right\rangle. \tag{3}$$

Here β is a symmetry parameter and is equal to 1, 2 or 4 for orthogonal, unitary and symplectic symmetries [5] respectively. The average is

$$\langle(\ldots)\rangle = Z^{-1} \int_0^\infty \cdots \int_0^\infty \left(\prod_{i=0}^{N-1} \mathrm{d}x_i\right) P\{x_0 \ldots x_{N-1}\}(\ldots) \tag{4}$$

and the partition function

$$Z = \int_0^\infty \cdots \int_0^\infty \left(\prod_{i=0}^{N-1} \mathrm{d}x_i\right) P\{x_0 \ldots x_{N-1}\}. \tag{5}$$

For simplicity we will consider $\beta = 2$ only, which corresponds to the case where a magnetic field is present, thereby breaking the time reversal symmetry.

We can think of equation (2) for the joint probability distribution as an exponential of some effective 'Hamiltonian' for the eigenvalues [5]. The first product term will then correspond to a logarithmic repulsion between the eigenvalues, which will try to separate them as far as possible, while the second product term will correspond to a single-particle 'confining potential' which keeps them from moving out to infinity.

3. Description of the model

The single-particle potential $V(x)$ defined in equation (2) was studied numerically in [3] in 2 and 3 dimensions for various sample sizes and disorders, and it was found that its functional form was essentially universal. In particular, a good fit was obtained by a function of the form

$$V(x) = a \ln^2(1 + bx) \tag{6}$$

where a and b depend on system parameters. This is a very unusual potential for which the associated random matrix model cannot be solved analytically for arbitrary n-point correlation functions. But this form for V is not at all surprising. It is known in one dimension [12] that, due to the multiplicative nature of the transfer matrices, the resistance distribution is log–normal. Since there is only one eigenvalue

in one dimension, the potential term entirely describes the distribution, and therefore must be of the form $[\ln x]^2$ for large x. In higher dimensions, the insulating regime will be qualitatively similar to the one-dimensional case and we expect the same functional form for $V(x)$ for large x. In the metallic regime (i.e. small eigenvalues) the fluctuation of the eigenvalues are known to obey Wigner–Dyson–Mehta statistics [5], so the potential must be of the usual polynomial form. Therefore such a potential arises quite naturally for transfer matrices related to disordered conductors.

It is clear that the $[\ln x]^2$ potential is only weakly confining compared to a linear or quadratic power law used in random matrix models, and it is plausible that the eigenvalue density as well as higher-order correlation functions may have features qualitatively different from the universal random matrix results mentioned in the introduction. In particular, because metallic behaviour is associated with well confined (small) eigenvalues and insulating behaviour with poorly confined ones, this opens up the possibility that the distribution of eigenvalues changes qualitatively from the metallic to the insulating regime, obtained by simply changing the parameters a and b in equation (6). It is therefore important to try to obtain predictions of such a model within our random matrix approach, especially for strong disorder. It should be possible to calculate, within a $1/N$ expansion [5, 6], at least some of the low-order moments of the distribution of eigenvalues. On the other hand, potential (6) is only an approximate two-parameter fit to numerical results [3] (more parameters could make the fit better), and there seem to be weak but subtle dimension-dependent corrections which apparently may not be obtained quantitatively within such a single-particle potential. It is not clear how important such corrections are for qualitative description of correlation functions in two or three dimensions. It is therefore important to obtain the consequences of a $[\ln x]^2$ potential (for large x) as accurately as possible, in order to understand any limitations of a single-particle model. We therefore choose to consider a qualitatively similar, though apparently more complicated, potential which is exactly solvable. Approximate calculations for more realistic models can then be tested against these exact solutions.

Consider the potential,

$$V(x) = -\ln[w(x;q)] \tag{7}$$

where

$$w(x;q) = 1/(-(1-q)x;q)_\infty \qquad 0 < q < 1 \tag{8}$$

and we have used the (conventional) notation

$$(a;q)_n = (1-a)(1-aq)\ldots(1-aq^{n-1}) \qquad (a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n). \tag{9}$$

(In [1] $w(x;q)$ was defined with an extra factor x^α, $\alpha > -1$. The model remains exactly solvable, but we will consider only the special case $\alpha = 0$ here to avoid unnecessary complexity.) Using the identity

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n}x^n = \frac{(ax;q)_\infty}{(x;q)_\infty} \tag{10}$$

known as the q-binomial theorem [13], w can be represented as a power series in x. Note that as $q \to 1^-$, $w \to e^{-x}$. We have explicitly written 1^- because the limit of

$q = 1$ has to be taken from below. In the following, we will always mean this limit when we consider $q = 1$. For sufficiently large x and small q, $V(x; q) \sim [\ln x]^2$. One simple way to see this would be to use an identity known as Jacobi's triple-product identity [13]:

$$(q; q)_\infty (-zq^{1/2}; q)_\infty \left(\frac{q^{1/2}}{-z}; q \right)_\infty = \sum_{n=-\infty}^{+\infty} q^{n^2/2} z^n. \tag{11}$$

By identifying

$$z = [(1-q)/q^{1/2}]x \tag{12}$$

and considering the limit $q \ll (1-q)x$ where the first and last factors in the triple product are of $O(1)$, the left-hand side of (11) can be identified with the infinite product of (8). The infinite sum on the right-hand side can be approximated by an integral using the Euler approximation to get the above behaviour. Thus the one-parameter potential V is qualitatively similar to (6), where q can be taken as a parameter such that decreasing q corresponds to increasing disorder. One can also introduce a scale factor t which multiplies x in the definition of w in equation (8). In this case, $q = 1$ means $w = e^{-tx}$, i.e. $V = tx$, while at $q = 0$, $V = \ln(1 + tx)$. The linear potential $V(x; 1)$ is known to describe the metallic regime very well [2, 14], with t as a disorder-dependent parameter. We will ignore the parameter t in the potential for simplicity, and include its effect at the end by appropriately scaling x in the definition of the conductance g, equation (1). The metallic region will then correspond to the case $q = 1$.

A family of orthogonal polynomials $P_n(x)$ can be defined with respect to a given measure [15] w:

$$\int_0^\infty dx \, w(x; q) P_m(x) P_n(x) = \delta_{mn} h_n. \tag{13}$$

It is then possible to express the eigenvalue density as well as higher-point correlation functions in terms of these polynomials [5]. Our model is solvable because for our particular choice (8) of w the polynomials are known explicitly. These are the q-Laguerre polynomials investigated by Moak [9]:

$$P_n(x) = L_n^{(0)}(x; q) = \sum_{k=0}^{n} \frac{(q^{-n}; q)_k q^{k(k-1)/2}(1-q)^k (q^{n+1}x)^k}{[(q; q)_k]^2}$$

$$h_n = \ln(1/q)/(1-q)q^n. \tag{14}$$

In terms of these polynomials the eigenvalue density is given as

$$\sigma_N(x; q) = w(x; q) \sum_{k=0}^{N-1} \frac{[L_k^{(0)}(x; q)]^2}{h_k} \tag{15}$$

with the obvious normalization

$$\int_0^\infty dx \, \sigma_N(x; q) = N. \tag{16}$$

4. The metallic regime

In the limit $q \to 1^-$, $L_N(x; q)$ reduces to the ordinary Laguerre polynomials, corresponding to the measure e^{-x}. In this limit, one finds

$$\sigma_N(x; 1) = Ne^{-x}[L_N(x)L'_{N-1}(x) - L_{N-1}(x)L'_N(x)] \tag{17}$$

where we have used the Christoffel–Darboux formula [14]

$$\sum_{n=0}^{N-1} \frac{P_n(x)P_n(y)}{h_n} = \frac{k_{N-1}}{k_N h_{N-1}} \frac{P_N(x)P_{N-1}(y) - P_{N-1}(x)P_N(y)}{x - y} \tag{18}$$

(k_N is the coefficient of x^N), and have taken the limit $x \to y$ to arrive at equation (17) from equation (15). Using an asymptotic expansion of the Laguerre polynomials [15] (valid for $x > 0$), namely

$$L_N(x) = e^{x/2} J_0\left(2\sqrt{(N + \tfrac{1}{2})x}\right) + O(N^{-3/4}) \tag{19}$$

the eigenvalue density becomes

$$\begin{aligned}
\frac{\sigma_N(x; 1)}{N} = (1/\sqrt{x}) &\left\{ \sqrt{N_+} J_0\left(\sqrt{4N_- x}\right) J_1\left(\sqrt{4N_+ x}\right) \right. \\
&\left. - \sqrt{N_-} J_0\left(\sqrt{4N_+ x}\right) J_1\left(\sqrt{4N_- x}\right) + O\left(N^{-3/2}\right) \right\}
\end{aligned} \tag{20}$$

where $N_\pm = N \pm \frac{1}{2}$ and $J_\nu(x)$ are the Bessel functions. We observe that in spite of the asymptotic form used, the normalization condition is fulfilled. Using this form for the eigenvalue distribution, the mean conductance $\langle g \rangle$ can now be determined:

$$\begin{aligned}
\frac{\langle g \rangle}{N} = \int_0^\infty \mathrm{d}x \, \frac{\sigma_N(x; 1^-)/N}{1 + x/t} &= 1 - \sqrt{4tN_-} I_1\left(\sqrt{4tN_-}\right) K_0\left(\sqrt{4tN_+}\right) \\
&- \sqrt{4tN_+} K_1\left(\sqrt{4tN_+}\right) I_0\left(\sqrt{4tN_-}\right)
\end{aligned} \tag{21}$$

where

$$I_\nu(x) \sim e^x/\sqrt{2\pi x})\{1 - (4\nu^2 - 1)/8x + \ldots\} \qquad (x \gg 1) \tag{22}$$

and

$$K_\nu(x) \sim \sqrt{(\pi/2x)}e^{-x}\{1 + (4\nu^2 - 1)/8x + \ldots\} \qquad (x \gg 1) \tag{23}$$

are the modified Bessel functions. From the asymptotic forms of $I_\nu(x)$ and $K_\nu(x)$ one finds, for $Nt \gg 1$,

$$\langle g \rangle/N \sim 1 - \tfrac{1}{2}\exp\left\{-\sqrt{4t}\left(\sqrt{N_+} - \sqrt{N_-}\right)\right\}[(N_-/N_+)^{1/4} + (N_+/N_-)^{1/4}]. \tag{24}$$

We can check that for $t \to \infty$, all eigenvalues are compressed at the origin giving $\langle g \rangle = N$, which is the ballistic limit. It is to be expected that in general, t will depend on physical parameters such as disorder, size and dimension. In the large N limit and for $Nt \gg 1$,

$$\langle g \rangle \sim N(1 - e^{-a}) \tag{25}$$

where $a = \sqrt{t/N}$. For $a \ll 1$, without violating $N \gg 1$ and $Nt \gg 1$, the exponential can be expanded to give the leading term Na, which suggests that we identify $a = l/L$, where l is the mean free path, thus reproducing Ohm's law given by

$$g_0 = N(l/L). \tag{26}$$

This identification agrees with an earlier saddle point calculation for the most probable conductance [14]. The next term is proportional to a^2, and it is tempting to identify it with the leading quantum correction to the conductance [4] $\delta g/g_0 = -\frac{1}{2}(l/L)$ in three dimensions, due to weak localization, where $Nt \gg 1$ is well satisfied. In two dimensions, in order to see the logarithmic size dependence, one needs to be in a limit where $\ln(L/l) \ll Nl/L$, $N \propto L$. This is outside our large-N limit. However, even in 3D, since we have restricted ourselves to the case $\beta = 2$, valid in a finite magnetic field, we should also identify a magnetic length in the problem. In order to be able to do that, we need to evaluate the average conductance for $\beta = 1$ (the case without a magnetic field) in terms of the same parameter and compare. The calculation for $\beta = 1$ involves somewhat more complicated 'skew' orthogonal polynomials [5], and constitutes a separate problem by itself. We plan to study it in the near future.

Note that for Laguerre polynomials the variance of g is known to be independent of N or t, giving rise to the universal conductance fluctuation [2]. This was the original motive for considering random matrix models for mesoscopic conductors.

5. The insulating regime

For $q < 1$, on the other hand, the qualitative features of the eigenvalue density changes dramatically. First of all, since $w(0;q) = 1$, we get, from (15)

$$\sigma_N(0; q < 1) = (1 - q^N)/\ln(1/q). \tag{27}$$

Compare this with the result for $q = 1$:

$$\sigma_N(0; q = 1) = N. \tag{28}$$

For large but finite N, the density at the origin drops from N to a value of order 1 as q changes from 1 to a value of order $1/N$. In the thermodynamic limit $N \to \infty$, the density at the origin drops to a value independent of N as soon as q is less than the critical value 1. In other words, for $q = 1$, increasing the number of eigenvalues corresponds to packing more of them at the origin, which, according to equation (1), may lead to a metal in the thermodynamic limit. For a given $q < 1$, density at the origin no longer scales with N, a feature qualitatively different from the standard random matrix ensembles [16]. This means that the conductance in this case will be at least a factor N less than the $q = 1$ case, which will make it non-metallic. Where do the eigenvalues go in this case as we increase N? We do not have an analytic solution yet for the eigenvalue density $\sigma_N(x; q \lesssim 1)$. In the following we show that for $q \ll 1$, the density has a gap from the origin up to order $1/q$ in the limit $N \to \infty$. Thus for exponentially small q, according to equation (1), this will correspond to an insulator. It is clear that a metal to insulator transition, if it is describable in the present model, would be associated with the sharp change in the scaling behaviour of the eigenvalue density near the origin at the critical value $q = 1$, and the associated opening of a gap.

In general, the q-Laguerre polynomials appearing in (14) for $q < 1$ have an asymptotic expansion (see appendix A) qualitatively different from (19) for $q = 1$. It is known in classical moment problems [17] that the measure w defined in (8) is not unique for $q < 1$, in the sense that there exist equivalent measures with exactly the same moments. Moak [9] has constructed a family of discrete (extreme, L^2–complete) measures supported at zeroes of an entire function, which is equivalent to our measure (8). It is considerably simpler to consider one such discrete measure, given by

$$w_{\text{dis}}(x;q) = \sum_{n=0}^{\infty} \omega_n \delta(x - \tau_n) \tag{29}$$

where τ_n are the zeroes of $L_\infty^{(0)}(x;q)$, $\omega_n = \omega(\tau_n)$ and

$$\frac{1}{\omega(x)} = \sum_{n=0}^{\infty} q^n [L_n^{(0)}(x;q)]^2$$

$$\equiv L_\infty^{(0)}(x;q)(\mathrm{d}/\mathrm{d}x)\left(x L_\infty^{(1)}(x;q)\right) - x L_\infty^{(1)}(x;q)(\mathrm{d}/\mathrm{d}x) L_\infty^{(0)}(x;q). \tag{30}$$

One may also use

$$L_\infty^{(\alpha)}(x;q) - L_\infty^{(\alpha)}(xq;q) = -x(1-q)q^{\alpha+1}L_\infty^{(\alpha+1)}(xq;q) \tag{31}$$

to show that

$$1/w_n = [1/(1-q)]L_\infty^{(0)}(\tau_n/q;q)[(\mathrm{d}/\mathrm{d}x)L_\infty^{(0)}(x;q)]_{x=\tau_n}. \tag{32}$$

By applying the above considerations to the eigenvalue density one finds

$$\sigma_\infty(x;q) = \lim_{N\to\infty} \sigma_N(x;q) = \frac{1-q}{\ln(1/q)} \sum_{n=0}^{\infty} \delta(x - \tau_n). \tag{33}$$

Therefore, in contrast to the $q = 1$ case, in the large-N limit the eigenvalue distribution depends in a rather sensitive way on the locations of the zeroes. From our previous discussion on the behaviour of the eigenvalue density at the origin, one could anticipate the qualitatively different behaviour of the density for $q = 1$ and $q < 1$ near the origin. It is clear that $\sigma_\infty(x;q)$ must have almost no support (a gap, starting from the origin) up to the first zero of $L_\infty^{(0)}(x;q)$. In appendix B we obtain the asymptotic form for $L_\infty^{(0)}(x;q)$ for sufficiently small q and large enough x such that $x(1-q) \gg \sqrt{q}$, from which it follows that the zeroes are located at

$$\tau_n \approx 1/(1-q)q^{2n+1} \qquad n = 0, 1, 2 \ldots. \tag{34}$$

This agrees with Moak's observation that the zeroes are very well separated. In order to proceed further we shall employ a parametric representation of the δ function,

$$\delta(x) = \frac{1}{\pi} \int_0^\infty \mathrm{d}t \, \cos xt \tag{35}$$

and approximate the sum over the zeroes in (33) by an integral to give

$$-\sigma_\infty(x;q) = \frac{(1-q)^2}{2\pi(\ln(1/q))^2} q \int_0^\infty \mathrm{d}t \{\cos[x(1-q)qt]\mathrm{Ci}(t) + \sin[x(1-q)qt]\mathrm{Si}(t)\} \tag{36}$$

where

$$\mathrm{Ci}(x) = -\int_x^\infty \frac{\mathrm{d}t}{t}\cos t \qquad \mathrm{Si}(x) = -\int_x^\infty \frac{\mathrm{d}t}{t}\sin t. \tag{37}$$

The eigenvalue density is found to be

$$\sigma_\infty(x;q) = \begin{cases} 0 & x(1-q) \leqslant 1/q \\ \dfrac{1-q}{2\beta^2 x} & x(1-q) > 1/q \end{cases} \tag{38}$$

which explicitly shows the gap up to $x = 1/q(1-q)$, and we have used β (not to be confused with the symmetry parameter defined in equation (2)) defined as in appendix B equation (65),

$$q = \mathrm{e}^{-\beta} \qquad 0 < \beta < \infty. \tag{39}$$

From the definition of $\langle g \rangle$, an elementary quadrature gives

$$\langle g \rangle(q) = \int_0^\infty \mathrm{d}x\, \frac{\sigma_\infty(x;q)}{1+\frac{x}{t}} = \frac{1-q}{2\beta^2}\ln[1+(\mathrm{e}^{-\beta} - \mathrm{e}^{-2\beta})t] \approx \frac{t}{2\beta^2}\mathrm{e}^{-\beta} \qquad \beta \gg 1. \tag{40}$$

Indeed by identifying $\beta = L/\xi$, where ξ can be identified with the localization length, we see that the conductance obtained in this regime describes an Anderson insulator.

We can try to understand the dramatic change between results for $q = 1$ and $q \ll 1$ in a qualitative way by considering the potential (7) in these two regimes. For $q = 1$ the potential is linear, which is well confining in the sense that it will always 'win' at sufficiently large distances over the logarithmic repulsion term in (2). For $q \ll 1$, the $[\ln x]^2$ potential for large x is only barely confining, allowing the eigenvalues to spread out to the point where it becomes discrete near the origin even in the thermodynamic limit, such that the smallest eigenvalue sits at an exponentially large distance. In order to convince the reader that the discreteness and hence the gap is necessarily a consequence of the $[\ln x]^2$ potential for large x and not a consequence of the discrete measure (29) chosen for convenience, we consider as an example the potential V associated with the log–normal measure,

$$w_s(x;q) = \left(1/\sqrt{\pi \ln(1/q^2)}\right)\exp(-(\ln x)^2/\ln(1/q^2)). \tag{41}$$

The associated polynomials (13) in this case are the Stieltjes–Wigert polynomials [15],

$$S_N(x;q) = (-1)^N \frac{q^{N/2+1/4}}{(q;q)_N^{1/2}} \sum_{k=0}^{N-1} \frac{(q;q)_N}{(q;q)_k (q;q)_{N-k}} q^{k^2}(-q^{1/2}x)^k. \tag{42}$$

In a saddle-point-type calculation [14], the eigenvalue distribution is found by summing over the zeroes λ_k of the Stieltjes–Wigert polynomials;

$$\sigma_s(x;q) = \sum_{k=1}^N \delta(x - \lambda_k). \tag{43}$$

We use the summation formula of Gauss [13], namely

$$\sum_{k=0}^N \frac{(q;q)_N}{(q;q)_k (q;q)_{N-k}} q^{k(k+1/2)}(-x)^k = \prod_{k=1}^N (1 - q^k x) \tag{44}$$

in (42) and find that the zeroes are located at

$$\lambda_k = 1/q^k \qquad k = 1,2,3\ldots,N. \tag{45}$$

For sufficiently large N, the eigenvalue density again has no support up to $1/q$ (has a gap), and can be shown to decrease as $1/x$ thereafter. These considerations confirm the characteristic features of a $[\ln x]^2$ potential.

6. The limit of one dimension

The resistance distribution $P(\varrho; q)$, for the one-dimensional problem ($N = 1$, i.e. with only one operating channel) in the large resistance limit, is simply

$$P(\varrho; q) = \int_0^\infty dx\, w(x; q) \delta\left(\frac{x}{t} - \varrho\right). \tag{46}$$

The large-resistance limit of $P(\varrho; q)$ can be found from the triple-product formula of Jacobi [13],

$$(q; q)_\infty (-zq^{1/2}; q)_\infty \left(\frac{q^{1/2}}{-z}; q\right)_\infty = \sum_{n=-\infty}^{+\infty} q^{n^2/2} z^n \tag{47}$$

by identifying

$$z = [(1 - q)/q^{1/2}]\varrho t. \tag{48}$$

For $q \ll (1 - q)\varrho t$, the first and last factors in the triple product are of O(1). We use the Euler approximation (equation (63), appendix B) to approximate the summation in the right-hand side by an integral to obtain

$$P(\varrho; q) \sim \left(1/\sqrt{\ln(q^{-1/2})}\right) \exp -\{[\ln(1 - q)\varrho t + \ln(q^{-1/2})]^2/4\ln(q^{-1/2})\} \tag{49}$$

in the region where $\ln(1 - q)x \approx \ln(q^{-1/2}) \gg 1$. With the identification $\ln(q^{-1/2}) \propto L/\xi$ as before where L is the length and ξ is the localization length of the system and by setting the scale factor $t \propto q$ (note that this does not affect our identification of the same parameter in the metallic regime, where $q = 1$), one recovers the log–normal distribution derived by Mel'nikov [12] from a heat equation in the hyperbolic plane.

At this point we should like to mention that although the discrete measure $w_{dis}(x; q)$ defined in (29) is quite distinct from the continuum measure w defined in (8), their asymptotics are in fact essentially the same. This can be seen from an explicit calculation of w_{dis}. The weight factor (32) of the discrete measure can be rewritten in the following form:

$$w_n^{-1} = [1/(1 - q)]\{(d/dx)[L_\infty^{(0)}(x/q; q)L_\infty^{(0)}(x; q)]\}_{x=\tau_n}. \tag{50}$$

The asymptotic condition, $x(1 - q) \gg \sqrt{q} \gg q^{3/2}$, allows us to employ equation (61), appendix A, for the asymptotic expansion under the derivative. Approximating the sum over zeroes by an integral, we arrive at

$$w_{dis}(x; q) \sim \left(1/\sqrt{\ln(q^{-1/2})}\right) \exp\{-[\ln(1 - q)x + \ln(q^{-1/2})]^2/4\ln(q^{-1/2})\}. \tag{51}$$

Therefore both the discrete and the continuum measures give the same resistance distribution in the stated asymptotic regions.

The moments of the resistance distribution may also be determined easily from either measure w_{dis} or w,

$$\langle \varrho^n \rangle = \int_0^\infty d\varrho\, \varrho^n w(\varrho; q) \bigg/ \int_0^\infty d\varrho\, w(\varrho; q) \equiv \int_0^\infty d\varrho\, \varrho^n w_{ex}(\varrho; q) \bigg/ \int_0^\infty d\varrho\, w_{ex}(\varrho; q)$$

$$= q^{-n(n+1)/2}(q; q)_n/(1 - q)^n \sim e^{n(n+1)L/\xi} \qquad (q \ll 1). \tag{52}$$

Although the exact forms of the moments are known, the distribution function cannot be uniquely reconstructed [17].

7. Summary and conclusion

We have proposed a solvable random matrix model for disordered conductors. The model is based on the assumption that a single-particle confining potential for the eigenvalues describes the relevant physical correlations at least qualitatively well. Physical as well as numerical considerations lead us to consider a novel kind of potential, where the associated random matrix properties show important deviations from known universal features of standard random matrix ensembles. We show that the density at the origin, which scales with the number of eigenvalues N for $q = 1$ as in the standard models, becomes essentially independent of N for large N when the disorder-related parameter q becomes less than unity. In the $N \to \infty$ limit, we show that this leads to the opening up of a gap in the eigenvalue density at the origin, of size $1/q$. We calculate the average conductance in the two regimes which allows us to identify this qualitative change in the eigenvalue density with the transition from metallic to insulating behaviour. (Note that in a Hamiltonian formulation, such transitions show up only at the density–density or current–current correlation functions, and the density of states does not show any critical behaviour; but in the transfer-matrix formulation the conductance is directly related to the eigenvalue density by equation (2).) However, a proper evaluation of the properties near the critical region $q \sim 1$ will have to wait till the large-N asymptotics of the q-Laguerre polynomials in this limit are known.

The dimension dependence in the present model comes entirely via the dimension dependence of the parameters t and q, perhaps through their dependence on N, namely $N \propto L^{d-1}$. On the insulating side, q is essentially independent of N, so insulators in all dimensions behave essentially in the same way. On the metallic side, the identification of t guarantees the correct dimension dependence of the Boltzmann conductance [14]. However, because these identifications are made in our $\beta = 2$ case without any reference to a magnetic length, and because there may be a subtle, though weak-dimension dependence not included in the single-particle potential as seen in the numerical evaluation of the variance of conductance [3], further work is necessary to understand whether the model includes dimension dependences in a natural way.

We have not yet evaluated the two-point correlation function or the associated level-spacing distribution, but certain qualitative statements can be made. For $q = 1$, the two-point correlation function decreases as a power law with the separation between the eigenvalues, for large separations. For $q < 1$, in the $N \to \infty$ limit, the positions of the eigenvalues are fixed, at the zeroes of some entire function, as is most easily seen from the discrete measure (29). Therefore we expect the two-point correlation function to go to a constant or oscillate at large distances, showing a long-range order. This should be a common characteristic of all such q-polynomials. Preliminary calculations on a q-Hermite measure, where much more is known about the associated polynomials [20], support this argument [21].

Appendix A. A complete asymptotic series for *q*-deformed generalized Laguerre polynomials

Recall that the generalized *q*-Laguerre polynomials are

$$L_n^\alpha(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n} \sum_{k=0}^n \frac{(q^{-n};q)_k q^{k(k-1)/2}[(1-q)xq^{\alpha+n+1}]^k}{(q^{\alpha+1};q)_k(q;q)_k}. \tag{53}$$

But

$$(q^{-n};q)_k = (1-q^{-n})\dots(1-q^{-n+k-1}) = (q^n-1)\dots(q^{n-k+1}-1)q^{\frac{k}{2}(-2n+k-1)}$$

$$= (-1)^k[(q;q)_n/(q;q)_{n-k}]q^{k(k-1)/2-kn}. \tag{54}$$

Thus, using $(q^{\alpha+1};q)_n = (q^{\alpha+1};q)_\infty/(q^{n+\alpha+1};q)_\infty$ we get

$$L_n^\alpha(x;q) = \frac{(q^{\alpha+1};q)_\infty}{(q^{\alpha+n+1};q)_\infty}\frac{1}{(q;q)_n}\sum_{k=0}^n\frac{(q;q)_n}{(q;q)_{n-k}}\frac{q^{k^2+\alpha k}}{(q;q)_k(q^{\alpha+1};q)_k}[x(q-1)]^k$$

$$= (q^{\alpha+1};q)_\infty\sum_{k=0}^n\frac{(q^{n-k+1};q)_\infty q^{k(k+\alpha)}}{(q;q)_\infty(q^{\alpha+n+1};q)_\infty(q;q)_k(q^{\alpha+1};q)_k}[x(q-1)]^k. \tag{55}$$

Now we use the *q*-binomial theorem [13] to get

$$\frac{(q^{n-k+1};q)_\infty}{(q^{\alpha+n+1};q)_\infty} = \sum_{j=0}^\infty\frac{(q^{-k-\alpha};q)_j}{(q;q)_j}q^{(\alpha+n+1)j}. \tag{56}$$

Thus

$$L_n^\alpha(x;q) = \frac{(q^{\alpha+1};q)_\infty}{(q;q)_\infty}\sum_{0\leqslant k\leqslant n,j\geqslant 0}\frac{(q^{-k-\alpha};q)_j}{(q;q)_j}q^{(\alpha+n+1)j}q^{k(k+\alpha)}\frac{[x(q-1)]^k}{(q;q)_k(q^{\alpha+1};q)_k}. \tag{57}$$

The above series is an explicit series and is also an asymptotic series. It gives the complete asymptotic expansions,

$$L_n^\alpha(x;q) = \frac{(q^{\alpha+1};q)_\infty}{(q;q)_\infty}\sum_{j=0}^\infty\frac{q^{j(\alpha+1+n)}}{(q;q)_j}\sum_{k=0}^n\frac{(q^{-k-\alpha};q)_j}{(q;q)_k}\frac{q^{k(k+\alpha)}}{(q^{\alpha+1};q)_k}[x(q-1)]^k. \tag{58}$$

Therefore,

$$L_n^\alpha(x;q) = \frac{(q^{\alpha+1};q)_\infty}{(q;q)_\infty}\left\{\sum_{k=0}^\infty\frac{(x(q-1))^k q^{k(k+\alpha)}}{(q;q)_k(q^{\alpha+1};q)_k}\right.$$

$$\left.+\frac{q^{\alpha+n+1}}{1-q}\sum_{k=0}^\infty\frac{(1-q^{-\alpha-k})q^{k(k+\alpha)}}{(q;q)_k(q^{\alpha+1};q)_k}[x(q-1)]^k + O(q^{2n})\right\}$$

$$\equiv L_\infty^\alpha(x;q) + [q^{\alpha+n+1}/(1-q)]L_\infty^\alpha(x;q)$$

$$- [q^{n+1}/(1-q)]L_\infty^\alpha(x/q;q) + O(q^{2n}) \tag{59}$$

where

$$L_\infty^{(\alpha)}(x;q) = \frac{(q^{\alpha+1};q)_\infty}{(q;q)_\infty} \sum_{k=0}^\infty \frac{q^{k^2+\alpha k}(1-q)^k(-x)^k}{(q;q)_k(q^{\alpha+1};q)_k}$$

$$\equiv [x(1-q)]^{-\alpha/2} J_\alpha^{(2)}[2\sqrt{x(1-q)};q] \qquad (60)$$

is an entire function of x.

Appendix B. Zeroes of $L_\infty^{(0)}(x;q)$

We can obtain an estimate of the locations of the zeroes of $L_\infty^{(0)}(x;q)$ for sufficiently small q and large enough x such that $x(1-q) \gg \sqrt{q}$, using the asymptotic expansion of the q-Bessel function [10],

$$J_\alpha^{(2)}(x;q) \approx \left(\frac{x}{2}\right)^\alpha \frac{(\sqrt{q};q)_\infty}{2(q;q)_\infty} [(iq^{(\alpha+1/2)/2}x/2;q^{1/2})_\infty$$

$$\times \sum_{k=0}^\infty \frac{(q^{\alpha+1/2};q)_k q^{k/2}}{(q;q)_k(iq^{(\alpha+1/2)/2}x/2;q^{1/2})_k} + (x \to -x)] \qquad (61)$$

for $\alpha = 0$. We keep only the first term in the expansion and use the following triple-product formula of Jacobi [13],

$$(q^{1/2};q^{1/2})_\infty(-zq^{1/4};q^{1/2})_\infty \left(\frac{q^{1/4}}{-z};q^{1/2}\right)_\infty = \sum_{n=-\infty}^{+\infty} q^{n^2/4}z^n \qquad (62)$$

with the choice $z_\pm = e^{\pm i\pi/2}\sqrt{x(1-q)}$, and notice that in the region of interest the first and last factors in the triple product are both of O(1). For $q \ll 1$, we may use the Euler approximation

$$\sum_{n=-\infty}^{+\infty} F(n) \approx \int_{-\infty}^{+\infty} dx\, F(x) \qquad (63)$$

to approximate the summation over n in the right-hand side of the triple-product formula to arrive at the following asymptotics

$$L_\infty^{(0)}(x;q) \sim \sqrt{4\pi/\beta} \cos(\pi y/\beta) \exp[(y^2 - \pi^2/4)/\beta] \qquad (64)$$

where

$$q = e^{-\beta} \qquad 0 < \beta < \infty \qquad y = \tfrac{1}{2}\ln[x(1-q)]. \qquad (65)$$

In this estimate the zeroes are found to coincide with the zeroes of $\cos(\pi y/\beta)$, and are at

$$\tau_n \approx 1/(1-q)q^{2n+1} \qquad n = 0, 1, 2 \ldots. \qquad (66)$$

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